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Compact analytical form for a class of three-loop vacuum Feynman diagrams

A.V. Kotikov

JINR, Dubna, 141980 Russia
E-mail: kotikov@sunse.jinr.ru

Abstract: We present compact, fully analytical expressions for singular parts of a class of three-loop diagrams which cannot be factorized into lower-loop integrals. As a result of the calculations we obtain the analytical expression for the three-loop effective potential of the massive φ^4 model presented recently by J.-M. Chung and B.K. Chung, *Phys. Rev.* **D 56** (1997) 6508.

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Though Quantum Field Theory already has a long history and a number of different approaches, Feynman diagrams (FD) are still the main source of its dynamical information. Vacuum (bubble) FD (without external momenta) considered here have several areas of applications. First of all, it is the evaluation of the effective potentials (for the recent study see [1] and references therein) and renormalization group characteristics (β-functions and anomalous dimensions) of quantum field models and specific operators (for example, anomalous dimensions of operators in the Wilson expansion). The second important area of application is the calculation of various processes (especially in Standard Model), where there is a possibility to neglect most of the masses and momenta and to calculate only some Taylor coefficients of multipoint FD. These coefficients are bubbles having different (sometimes quite big) powers of propagators (i.e. indices of propagators). Using different recurrence relations [2, 3, 4] it is possible usually to represent these Taylor coefficients as sets of very simple (usually one-loop) bubbles and several so-called master integrals, which cannot be factorized into sums of lower-loop integrals. These recurrence relations are some particular cases of the relation [5, 6] for a general n-point (sub)graph with masses of its lines $m_1, m_2, ..., m_n$, line momenta $p_1, p_2 = p_1 - p_{12}, p_n = p_1 - p_{1n}$ and indeces $j_1, j_2, .., j_n$, respectively,

$$
\begin{aligned}
0 &= \int d^D p_1 \frac{\partial}{\partial p_1^\mu} \Bigl(\prod_{i=1}^{n} c_i^{j_i} \Bigr)^{-1} \qquad (1) \\
&= \int d^D p_1 \Bigl(\prod_{i=1}^{n} c_i^{j_i} \Bigr)^{-1} \Biggl[D - 2j_1 \Bigl(1 - \frac{m_1^2}{c_1} \Bigr) - \sum_{i=2}^{n} j_i \Bigl(1 - \frac{m_1^2 + m_i^2 + p_{1i}^2 - c_1}{c_i} \Bigr) \Biggr],
\end{aligned}
$$

where $c_k = p_k^2 + m_k^2$ are the propagators of n-point (sub)graph. The equation (1) is based on the rule of integration by part [7] and has been discovered in the process of the calculation of propagator- and vertex-type diagrams (in [5] for the case $n = 3$) and of n-point diagrams (in [6] for arbitrary n).

Since the diagram with the index $(i+1)$ of the propagator c_i may be represented as the derivative (in the mass m_i), eq. (1) leads to the differential equations (in principle, to partial differential equations) for the initial diagram (having the index i, respectively). This approach which is based on eq. (1) and allows to construct the (differential) relations between diagrams has been named Differential Equations Method (DEM).[1]. For most interesting cases (where the number of the masses is limited) these partial differential equations may be represented through ordinary differential equation[2], which is usually simpler to analyse. For the bubbles, DEM has been used in [10, 11].

In the case of large number of diagrams (for example, in the calculation of various processes of Standard Model) it is convenient [3, 4] to use eq. (1) without representing the diagrams in its r.h.s. as derivatives of initial ones. Then eq. (1) gives the connections between different diagrams reducing essentially the number of complicated integrals. These integrals (i.e. master diagrams) may be calculated [12] with help of DEM as it was discussed above.

[1]A similar technique has been introduced [8] recently by Remiddi

[2]The example of the direct usage of the partial differential equation may be found in [9].

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1. In the present article we consider a class of three-loop bubbles (see Fig. 1) having two different masses: the masses of thin lines are one-third of the masses of the bold lines. The diagrams are interesting because they cannot be factorized into lower-loop integrals (see discussions in [13]). Moreover, some of them ($J(a)$, $K(a)$, $L(a)$ and $M(a)$) give contributions (see [1, 13]) to the three-loop effective potential of the massive $O(N)$ φ^4-model. The diagrams have been studied recently in [13] using the methods developed in the articles [14, 15].[3] The following results for their singular parts have been found[4]:

$$
\begin{aligned}
J(a) &= N_2\Big[\frac{2}{\varepsilon^3} + \frac{23}{3}\frac{1}{\varepsilon^2} + \frac{35}{2}\frac{1}{\varepsilon} + O(1)\Big] \\
J(b) &= N_2\Big[\frac{22}{27}\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2}\Big(\frac{83}{27} + \frac{7}{9}\log 3\Big) + \frac{1}{\varepsilon}\Big(\frac{365}{54} + \frac{55}{18}\log 3 + \frac{1}{2}\log^2 3\Big) + O(1)\Big] \\
J(c) &= N_2\Big[\frac{2}{9}\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2}\Big(\frac{23}{27} + \frac{2}{3}\log 3\Big) + \frac{1}{\varepsilon}\Big(\frac{35}{18} + \frac{23}{9}\log 3 + \log^2 3\Big) + O(1)\Big] \\
K(a) &= -N_1\Big[\frac{1}{\varepsilon^3} + \frac{17}{3}\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(\frac{67}{3} + 6A\Big) + O(1)\Big] \\
K(b) &= -N_1\Big[\frac{7}{9}\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2}\Big(\frac{13}{3} + \frac{1}{3}\log 3\Big) + \frac{1}{\varepsilon}\Big(\frac{151}{9} + 3A + \frac{1}{3}B + 2\log 3\Big) + O(1)\Big] \\
K(c) &= -N_1\Big[\frac{5}{9}\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2}\Big(\frac{29}{9} + \frac{2}{3}\log 3\Big) + \frac{1}{\varepsilon}\Big(\frac{37}{3} + \frac{2}{3}B + 4\log 3\Big) + O(1)\Big] \qquad (2) \\
K(d) &= -N_1\Big[\frac{5}{9}\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2}\Big(3 + \frac{2}{3}\log 3\Big) + \frac{1}{\varepsilon}\Big(\frac{101}{9} + \frac{2}{3}B + 4\log 3\Big) + O(1)\Big] \\
L(a) &= N_0\Big[\frac{1}{3}\frac{1}{\varepsilon^3} + \frac{2}{3}\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(\frac{2}{3} + 2A\Big) + O(1)\Big] \\
L(b) &= N_0\Big[\frac{1}{3}\frac{1}{\varepsilon^3} + \frac{2}{3}\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(\frac{2}{3} + A + C\Big) + O(1)\Big] \\
L(c) &= N_0\Big[\frac{1}{3}\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2}\Big(\frac{2}{3} + \log 3\Big) + \frac{1}{\varepsilon}\Big(\frac{2}{3} + 2B + 2\log 3 + \frac{3}{2}\log^2 3\Big) + O(1)\Big] \\
L(d) &= N_0\Big[\frac{1}{3}\frac{1}{\varepsilon^3} + \frac{2}{3}\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(\frac{2}{3} + 2C\Big) + O(1)\Big] \\
M(a) &= M(b) = M(c) = N_0\Big[\frac{2}{\varepsilon}\zeta_3 + O(1)\Big],
\end{aligned}
$$

where the normalization factor

$$N_k = \frac{m^{2k}}{(4\pi)^6}\Big(\frac{m^2 e^{\gamma}}{4\pi\mu^2}\Big)^{-3\varepsilon} \cdot \exp\left(3/2\zeta_2\varepsilon^2\right)$$

and ζ_n are Euler ζ-functions.

The constants A, B and C have been represented [13] in the form

$$A = f(1,1), \quad B = f(1,3) \quad \text{and} \quad C = f(\frac{1}{3},\frac{1}{3}),$$

[3]The singular part of $M(a)$, $M(b)$ and $M(c)$ diagrams has been calculated in [11] using DEM [5]. The regular part of $M(a)$ has been found very recently by Broadhurst [16].

[4]Contrary to [13] we use the space $D = 4 - 2\varepsilon$ and sum the terms γ^n (γ - is Euler constant) and ζ_2^n to exponents.

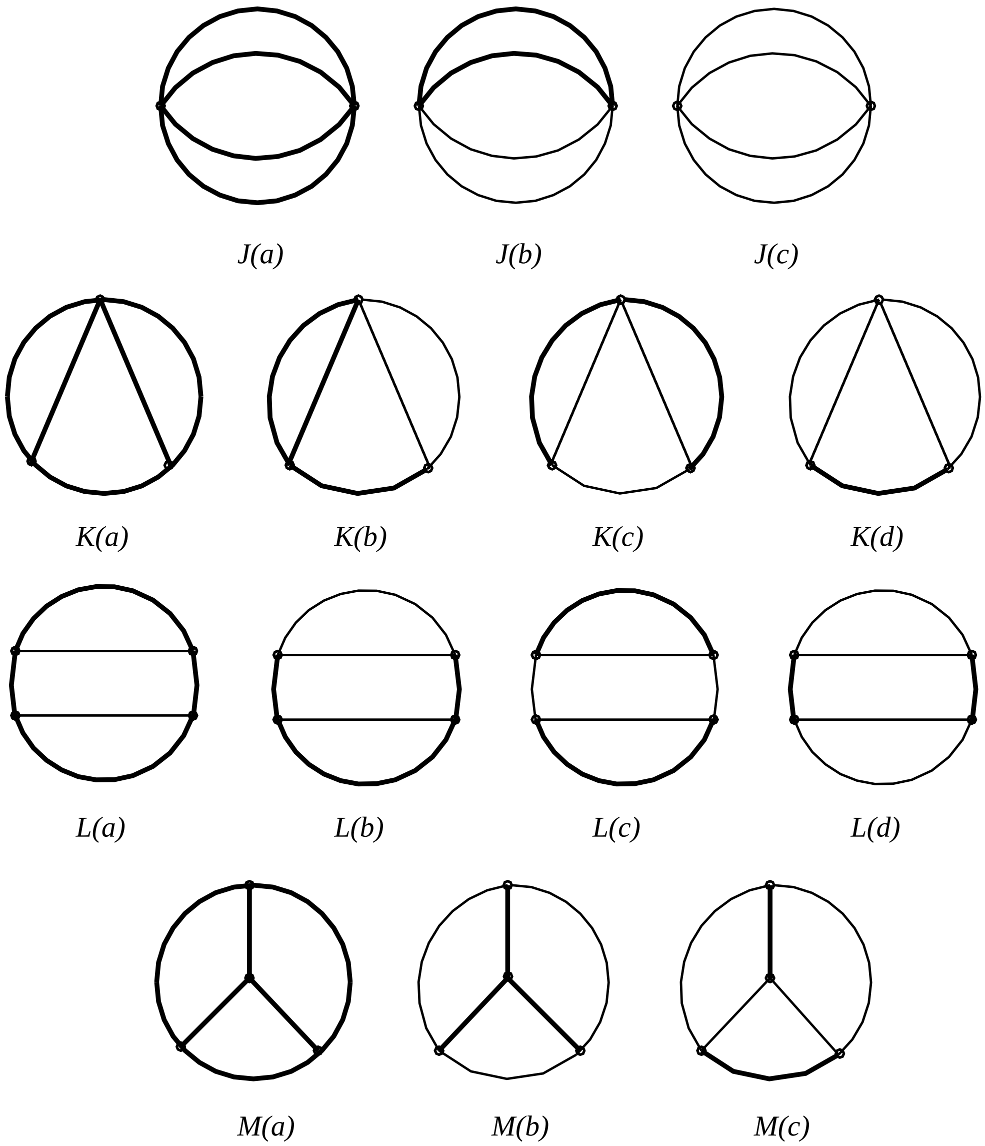

Figure 1: The class of three-loop bubbles. The masses of thin lines are one-third of the masses of the bold lines.

where

$$f(a,b) = \int_0^1 dx \left[\int_0^{1-z} dy \left(-\frac{\log(1-y)}{y} \right) - \frac{z \log z}{1-z} \right], \quad z = \frac{ax + b(1-x)}{x(1-x)}. \tag{3}$$

The purpose of this short paper is to calculate analytically A, B and C constants and, thus, to obtain exact results for the singular parts of the FD presented in Fig. 1.

Before evaluations we would like to stress that the knowledge of the exact values for the singular parts of diagrams is very important, because they determine effective potentials and renormalization functions of the quantum field models. Moreover, for various physical processes, the regular parts of diagrams may be evaluated numerically (sometimes with rather qood accuracy) but their singular parts should be known analytically because many types of them should be cancelled at the end of calculations.

2. To evaluate $A = f(1,1)$ we represent the r.h.s. of eq. (3) as the sum of two terms A_1 and A_2.

We introduce the new variable $s = (1-x)/2$ and represent A_2 as

$$A_2 \equiv -\int_0^1 dx \frac{z \log z}{1-z} = -4\int_0^1 \frac{ds}{3+s^2} \log\Big(\frac{1-s^2}{4}\Big).$$

The term A_1 may be rewritten in the form $(y = 1-t)$

$$A_1 \equiv \int_0^1 dx \int_0^{1-z} dy \frac{\log(1-y)}{y} = \int_0^1 ds \int_1^{4/(1-s^2)} dt \frac{\log t}{1-t}. \tag{4}$$

Changing the order of integration in the r.h.s. of (4), we have

$$A_1 = 6\int_0^1 \frac{ds}{3+s^2} \log\Big(\frac{1-s^2}{4}\Big).$$

Thus, the evaluation of

$$A = 4\int_0^1 \frac{ds}{3+s^2} \Big[\log(1-s) + \log(1+s) - \log 4\Big]$$

is very simple. Integrating by part, we obtain the final result

$$A = -\frac{2}{\sqrt{3}} \mathrm{Cl}_2\Big(\frac{\pi}{3}\Big), \tag{5}$$

where $\mathrm{Cl}_2(\theta)$ is Clausen integral [17]

$$\mathrm{Cl}_2(\theta) = \int_0^\theta \log(2\sin(\theta'/2)) d\theta'.$$

Repeating the above calculations, we have

$$B = -\frac{4}{\sqrt{3}} \mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) \quad \text{and} \quad C = -\frac{1}{2}\log^2 3 + \frac{4}{3\sqrt{3}} \mathrm{Cl}_2\Big(\frac{\pi}{3}\Big). \tag{6}$$

3. We have obtained results for the singular parts of a FD class in closed analytical form. These analytical results are important in the calculations of physical processes of Standard Model, where many terms may be involved and the verification and the evaluation of the singularities is very important problem. Moreover, the constant A is the only numerical factor in the recent calculation [1] of three-loop correction to the effective potential of massive φ^4 model:

$$\begin{aligned}
-\frac{(4\pi)^6}{\lambda^2} \cdot V^{(3)}_{eff}(\varphi_c) = {} & \frac{\lambda^2\varphi_c^4}{4}\Big\{\frac{1129}{192} + \frac{A}{8}\Big\} + \frac{m^2\lambda\varphi_c^2}{2}\Big\{-\frac{25}{96} - \frac{A}{4}\Big\} + \\
& + \Bigg[\frac{\lambda^2\varphi_c^4}{4}\Big\{-\frac{629}{96} - \frac{3A}{4} - \zeta_3\Big\} + \\
& \quad + \frac{m^2\lambda\varphi_c^2}{2}\Big\{-\frac{287}{48}\Big\} + m^4\Big\{\frac{25}{96} + \frac{A}{4}\Big\}\Bigg] \cdot \log\Big(1 + \frac{\lambda\varphi_c^2}{2m^2}\Big) + \\
& + \Bigg[\frac{\lambda^2\varphi_c^4}{4} \cdot \frac{143}{48} + \frac{m^2\lambda\varphi_c^2}{2} \cdot \frac{17}{6} + m^4 \cdot \frac{11}{48}\Bigg] \cdot \log^2\Big(1 + \frac{\lambda\varphi_c^2}{2m^2}\Big) - \\
& - \Bigg[\frac{\lambda^2\varphi_c^4}{4} \cdot \frac{9}{16} + \frac{m^2\lambda\varphi_c^2}{2} \cdot \frac{7}{12} + m^4 \cdot \frac{5}{48}\Bigg] \cdot \log^3\Big(1 + \frac{\lambda\varphi_c^2}{2m^2}\Big).
\end{aligned}$$

After the above-mentioned calculation of A (i.e. eq. (5)), this three-loop correction becomes known completely analytically.

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Note added. After submittion of this paper to hep-ph, the results (5) and (6) have been used already in [18] for the calculation of the effective potential of $O(N)$ φ^4 model.

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